B.Sc. Semester-III Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 32112 **Course Code : SH/MTH/302/C-6**

Course Title : Group Theory-I

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT–I

1. Answer any **five** of the following questions: 2×5=10

a) Does $\mathbb{R}$ (the set of all real numbers) form a group with respect to the binary operation $*$ defined by $a*b=2^a b$ for all $a, b \in \mathbb{R}$? Justify your answer.

b) Write down the complete Cayley table for the Dihedral group D_3 (the group of all symmetries of an equilateral triangle).

c) Find the order of the element $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 1 & 2 & 3 \end{pmatrix}$ in S_6.

d) Lt $(G, \bullet)$ be a group and $a \in G$ is an element of finite order. Let $a^m = e$ (where e is the identity element of the group G). Then prove that order of a must divide m.

e) Write down all non-trivial proper normal subgroups of the group $(\mathbb{Z}_6, +)$.

f) Find the order of H if H is a subgroup of some group of order 100 and H contains no element of order 2, with H is non-cyclic.

g) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic of order 8.

h) Let G_1 be a non-commutative group $\phi: G_1 \to G_2$ be an epimorphism and $|G_2| \geq 2$. Is G_2 non-commutative always? Give proper logic in support of your answer.

UNIT–II

2. Answer any **four** of the following questions: 5×4=20

a) In the permutation group S_4, exhibit explicity, with clarification, a cyclic subgroup H of order 4 and a non-cyclic subgroup K of order 4. 2+3=5

b) i) Prove that $(\mathbb{Q}, +)$ is not cyclic.

ii) Consider the group $(\mathbb{Q}/\mathbb{Z}, +)$. Prove that for any natural number n, there exists an element of the group of order n. 3+2

c) Let $f: G \to H$ be an onto homomorphism. Prove that $\frac{G}{Ker\, f} \cong H$.

d) Define normal subgroup of a group. Prove that $H = \{A \in GL(2, \mathbb{R}) : |A| \in K\}$ is a normal subgroup of the group $GL(2, \mathbb{R})$ where K is a subgroup of the multiplicative group $\mathbb{R}^*$ of all non-zero real numbers. 1+4=5

e) If in a group G, $a^5 = e$, $aba^{-1} = b^2$, then find $O(b)$.

f) If $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ and $H = \left\{\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} : a, b \in \mathbb{Q}\right\}$, show that G and H are isomorphic under addition.

UNIT–III

3. Answer any **one** of the following questions: 10×1=10

a) i) Let G be a group with the property: "for any $x, y, z \in G$, $xy = zx$ implies that $y = z$."

Prove that G is abelian.

ii) In the symmetric group S_n, prove that the number of even permutations is same as the number of odd permutations.

iii) Let H be a normal subgroup of a finite group G and $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$.

iv) Suppose that there is a homomorphism ϕ from $(\mathbb{Z}_{17}, +)$ to some group G such that ϕ is not one-one. Determine ϕ.

2+4+2+2=10

b) i) Compute (with proper clarification) the center of the dihedral group D_4.

ii) Justify whether $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \times)$ are isomorphic, where $(\mathbb{R}, +)$ is the group of all real numbers under addition and $(\mathbb{R}^+, \times)$ is the group of all positive real numbers under multiplication.

iii) Prove that $2\mathbb{Z}/16\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}/16\mathbb{Z}$. Hence find the factor group $(\mathbb{Z}/16\mathbb{Z})/(2\mathbb{Z}/16\mathbb{Z})$. 3+3+(2+2)=10

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